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Families of separated sets ☆

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Abstract

We characterize the possible subsets of $\mathcal{P}(\kappa)$ which can arise as the family of separated subsets of a closed discrete subset κ of a topological (Hausdorff, regular or normal) space.

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A discrete family of points D (equivalently, a closed discrete set) of a topological space X is said to be *separated* if there is a disjoint family $\{U_d\colon d \in D\}$ of open sets in X such that $(\forall d \in D)\ d \in U_d$.

If we know in a topological, Hausdorff, regular or normal space that certain subsets of a discrete family of points are separated, then what other subsets can we deduce to be separated simply from that information? In Hausdorff spaces, we know that finite subsets are separated. In regular spaces, we know that adding a finite set to a separated subset yields another separated set and we also know that countable sets are separated. In normal spaces, we know that the union of two separated sets is separated. What other deductions can be made?

If X is a topological space and D is a discrete family of points, then we can form $\mathcal{I}$, the family of separated subsets of D. We can then forget the topological setting of $\mathcal{I}$ and simply view $\mathcal{I}$ as a family of subsets of a fixed set D which might as well be a cardinal.

This article is devoted to the investigation of the possible subsets of $\mathcal{P}(D)$ which can be obtained in this way from topological spaces which satisfy various separation properties such as Hausdorff, regular or normal. We can completely characterize the possible subsets of $\mathcal{P}(D)$ which can arise from topological spaces, Hausdorff spaces, regular spaces and normal spaces. The question of obtaining such a characterization for

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normal spaces occurred to us after we heard Problem 1 below. After we announced the solution to that question, the question for regular spaces was posed by Zoltan Balogh.

We should emphasize that one cannot reduce the problem for regular spaces to the problem for topological spaces by taking a topological space which exhibits a certain $\mathcal{I}$ and, then, declaring just enough sets to be open so that the resulting space is regular. The reason for this is that it is impossible to control the deductions which become possible in the larger topology without a fine analysis of the deductions which are possible in a topological space.

To solve this problem for normal spaces, we need the concept of a nonprincipal ideal. If $\mathcal{I}$ is a subset of $\mathcal{P}(\kappa)$ which is closed under finite unions and is also closed under taking subsets, then we say that $\mathcal{I}$ is an *ideal*. If $\mathcal{I}$ contains all finite sets, then we say that $\mathcal{I}$ is non-principal. Intuitively, $\mathcal{I}$ is a notion of "smallness". Thinking measure-theoretically, we use $\mathcal{I}^+$ to indicate the "sets of positive measure" with respect to $\mathcal{I}$. That is, we say $A \in \mathcal{I}^+$ if $A \notin \mathcal{I}$. Of course, we are primarily interested in proper ideals on κ, that is, usually $\kappa \notin \mathcal{I}$.

Theorem 1. *Let $\mathcal{I}$ be a subset of $\mathcal{P}(\kappa)$. The following are equivalent.*

(1) *There is a normal space X with a discrete family of points κ such that $\{A \subset \kappa\colon A$ is separated$\} = \mathcal{I}$.*

(2) *There is a perfectly normal space X with a discrete family of points κ such that $\{A \subset \kappa\colon A$ is separated$\} = \mathcal{I}$.*

(3) *$\mathcal{I}$ is a nonprincipal σ-ideal.*

To prove this lemma, we need a Δ-system lemma for countable families of finite sets.

Lemma 1. *If $\{F_n\colon n \in \omega\}$ is a family of finite sets of bounded size, then there is an infinite $E \subset \omega$ and a finite set Δ such that $(\forall n, m \in E)$ $F_n \cap F_m = \Delta$.*

Proof. Assume that there is $k \in \omega$ such that, for each $n \in \omega$, $F_n = \{f_i^n\colon i \in k\}$. Define $\pi : [\omega]^2 \to 2^{k^2}$ by $\pi(\{m,n\})(i,j) = 1$ if and only if $f_i^m = f_j^n$ where $m < n$. By Ramsey's theorem, we can find some infinite $E \subset \omega$ such that π is constant on $[E]^2$. A simple argument shows that $(i,j) \in \pi([E]^2) \Rightarrow i = j$. Choose $a_0, a_1 \in E$ and let $\Delta = \{f_i^{a_0}\colon f_i^{a_0} = f_i^{a_1}\}$.

Proof of Theorem 1. (1) $\Rightarrow$ (3) If $A \in \mathcal{I}$ and $A' \subset A$, then $A' \in \mathcal{I}$. An application of ω-collectionwise normality shows that $\mathcal{I}$ is closed under countable union. Each one element subset of A is separated.

(3) $\Rightarrow$ (2) Let

$$\begin{aligned}\mathrm{ISO} = \big\{\{(\alpha, G, V), (\beta, H, W)\} \in \big[\kappa \times [\mathcal{P}(\kappa)]^{<\omega} \times [\mathcal{I}]^{<\omega}\big]^2\colon \\ (\forall A \in G \cap H)\ \alpha \in A \Leftrightarrow \beta \in A; (\forall P \in V \cap W)\{\alpha, \beta\} \not\subset P\big\}.\end{aligned}$$

Topologize $X = \kappa \cup \mathrm{ISO}$ by letting ISO consist of isolated points and letting a neighborhood $\alpha(F, U)$ of $\alpha \in \kappa$ in parameters $F \in [\mathcal{P}(\kappa)]^{<\omega}$ and $U \in [\mathcal{I}]^{<\omega}$ be

$$\{\alpha\} \cup \big\{\{(\alpha, G, V), (\beta, H, W)\}\colon G \supset F,\ V \supset U\big\}.$$

Since $F \subset F'$, $U \subset U'$ implies $\alpha(F,U) \supset \alpha(F',U')$ and F, U can be arbitrary finite subsets of $\mathcal{P}(\kappa)$ and $\mathcal{I}$ respectively and since the intersection of $\alpha(F,U)$ and $\alpha'(F',U')$ consists of isolated points when $\alpha \neq \alpha'$, we know that this defines a topological space. To show that X is T_1, it suffices to note that

$$\bigcap\left\{\alpha(F,U)\colon\ F \in \left[\mathcal{P}(\kappa)\right]^{<\omega},\ U \in [\mathcal{I}]^{<\omega}\right\} = \{\alpha\}.$$

To show that X is normal, it is standard to note that it suffices to find disjoint open sets around disjoint subsets A, A' of κ [3]. The open sets $\bigcup\left\{\alpha(\{A\},\emptyset)\colon\ \alpha \in A\right\}$ and $\bigcup\{\alpha(\{A\},\emptyset)\colon\ \alpha \in A'\}$ work. To show that X is perfect, it suffices to note that if $A \subset \kappa$ and $\{J_n\colon\ n \in \omega\} \subset \mathcal{I}$ is an infinite set, then

$$\bigcap\left\{\bigcup\left\{\alpha(F,\{J_i\colon\ i \in n\})\colon\ \alpha \in A\right\}\colon\ n \in \omega\right\} = A.$$

If R is in $\mathcal{I}$, then $\{\alpha(\emptyset,\{R\})\colon\ \alpha \in R\}$ is a separation of R. The hard part of the theorem is to show that if R is not in $\mathcal{I}$, then R is not a separated subset of κ in X. To do this, we suppose $R \in \mathcal{I}^+$ and that F_α and U_α have been defined so that $\{\alpha(F_\alpha, U_\alpha)\colon\ \alpha \in R\}$ is a disjoint family of open sets. Let $n : R \to \omega$ be defined by $n(\alpha) = |F_\alpha|$. Find $D \in \mathcal{I}^+$ and $n \in \omega$ such that $(\forall \alpha \in D)\ n(\alpha) = n$. Choose $\{\alpha_n\colon\ n \in \omega\} \subset D$ (using the fact that $\mathcal{I}$ is nonprincipal) such that $\alpha_n \notin \bigcup\{\bigcup U_{\alpha_i}\colon\ i < n\}$. Apply Lemma 1 to assume, without loss of generality, that $\{F_{\alpha_n}\colon\ n \in \omega\}$ is a Δ-system with root Δ. Find $i, j \in \omega$ such that $(\forall A \in \Delta)\ \alpha_i \in A \Leftrightarrow \alpha_j \in A$. We claim that $\alpha_i(F_{\alpha_i}, U_{\alpha_i}) \cap \alpha_j(F_{\alpha_j}, U_{\alpha_j}) \neq \emptyset$. In fact, we need only show that $\{(\alpha_i, F_{\alpha_i}, U_{\alpha_i}), (\alpha_j, F_{\alpha_j}, U_{\alpha_j})\} \in \mathrm{ISO}$.

If $A \in F_{\alpha_i} \cap F_{\alpha_j}$, then $A \in \Delta$ and so $\alpha_i \in A \Leftrightarrow \alpha_j \in A$. Suppose $P \in U_{\alpha_i} \cap U_{\alpha_j}$. We can assume $i < j$. Since $\bigcup\{\bigcup U_{\alpha_i}\colon\ i < j\} \supset P$, the inductive choice means that $\alpha_j \notin P$ and so $\{\alpha_i, \alpha_j\} \not\subset P$ as required. $\square$

A corollary to Theorem 1 answers the following problem of Gary Gruenhage.

Problem 1 (Gruenhage). Is there a normal space which has an unseparated discrete family of points but in which there do not exist two disjoint unseparated discrete families of points?

Corollary 1. *There is a normal space which has an unseparated discrete family of points but in which there do not exist two disjoint unseparated discrete families of points if and only if there is a measurable cardinal.*

Proof. If $\mathcal{I}^+$ does not contain two disjoint sets, then, for any $A \subset \kappa$, either $A \in \mathcal{I}$ or $\kappa - A \in \mathcal{I}$. Thus the dual of $\mathcal{I}$ is a nonempty nonprincipal countably complete ultrafilter.

Now we turn our attention away from normal spaces and so $\mathcal{I}$ will often not be an ideal.

Definition 1. If $\mathcal{I} \subset \mathcal{P}(\kappa)$, then we say $\mathcal{I}$ is *diagonally-closed* if, whenever $A \subset \kappa$, $\{E_i\colon\ i \in \omega\}$ is an increasing family of finite subsets of $\mathcal{I}$ and $\{B_i\colon\ i \in \omega\}$ is a partition of A such that

$$(\forall i \in \omega)(\forall \beta \in B_i)(\forall \gamma \in A)(\exists C \in E_i)\ \{\beta, \gamma\} \subset C$$

then, $A \in \mathcal{I}$. Note that diagonally closed implies closed under subsets but does not imply closed under finite unions.

To prove the next theorem, we need to use the notion of a finite hull in our argument. These are just the finite approximations to countable elementary submodels which occur in the usual proof of the Löwenheim–Skolem theorem for a fixed formula (i.e., in the proof of the Lévy reflection principle).

Proposition 1. *Let $\phi(x, v_0, \ldots, v_n)$ be a formula of set theory with free variables x and the v_i's. If E is any finite set, then there is a sequence of finite sets $\{\mathcal{M}_i\colon\ i \in \omega\}$ such that*

(1) $\mathcal{M}_0 \supset E$.

(2) $(\forall i \in \omega)(\forall m_0, \ldots, m_n \in \mathcal{M}_i)$
$((\exists x)\phi(x, m_0, \ldots, m_n) \Rightarrow (\exists x \in \mathcal{M}_{i+1})\phi(x, m_0, \ldots, m_n))$
(we call this "reflection").

(3) $\mathcal{M}_i \in \mathcal{M}_{i+1}$.

(4) $(\forall n \in \omega)(\forall F \in \mathcal{M}_n)(|F| < \aleph_0 \Rightarrow F \subset \mathcal{M}_n)$
(we call this "finitary transitivity").

We can also find a single sequence which works for finitely many formulas $\phi_0, \ldots, \phi_m$ simultaneously.

We call $\{\mathcal{M}_i\colon\ i \in \omega\}$ a sequence of finite hulls *for ϕ (for $\phi_0, \ldots, \phi_n$) and E.*

Proof. Simply construct $\{\mathcal{M}_i\colon\ i \in \omega\}$ by induction, observing that there are only, for each $i \in \omega$, finitely many "reflection" requirements that certain x's be in $\mathcal{M}_{i+1}$. The "finitary transitivity" requirement can be accomplished because of the next lemma. □

Lemma 2. *If A is a finite set, then there is a finite set B such that $B \supset A$ and $(\forall n \in \omega)(\forall F \in B)\ |F| < \aleph_0 \Rightarrow F \subset B$.*

Proof. Construct a tree T whose root is A. If C is a node in T, then the successors of C in T are the elements of C if C is a finite set. The node C has no successors if C is infinite. If the tree is finite, then the union of the nodes of T is the required finite set B. If the tree is infinite, then by König's lemma, there is an infinite branch. But an infinite branch contradicts the well-foundedness of the $\in$ relation. □

Proposition 2. *If $m_0, \ldots, m_n \in \mathcal{M}_i$, $\phi(x, v_0, \ldots, v_n)$ is a formula of set theory and there is a unique x_0 such that $\phi(x_0, m_0, \ldots, m_n)$ is true, then $x_0 \in \mathcal{M}_{i+1}$. Thus any set definable from parameters in $\mathcal{M}_i$ is an element of $\mathcal{M}_{i+1}$.*

Proposition 3. *$\mathcal{M}_n$ is a subset of $\mathcal{M}_{n+1}$ and, for a sufficiently large list of ϕ's,*[1] *every subset of $\mathcal{M}_n$ is an element of $\mathcal{M}_{n+2}$.*

[1] That is, provided that the set $\phi_1, \ldots, \phi_n$ contains a certain finite set of formulas.

Proof. The first statement follows from conditions (3) and (4). Choose ϕ which uniquely defines the power set. Since, by condition (3), $\mathcal{M}_n \in \mathcal{M}_{n+1}$, we have, by condition (2), $\mathcal{P}(\mathcal{M}_n) \in \mathcal{M}_{n+2}$. Now condition (4) gives the result. $\square$

Proposition 1 does not use the model-theoretic notions of satisfaction, truth or models of set theory but it does use the notion of a formula. However, by substituting any fixed formula ϕ into this proposition, we obtain a purely mathematical proposition which does not even involve the notions of formula or free variable. So although Proposition 1 is metamathematical, it is a scheme for infinitely many mathematical propositions. Of course, once a particular ϕ is chosen, the proof of that instance of Proposition 1 is also purely mathematical and involves no metamathematical notions. For more details, see [1]. In practice, at any time in a proof, a sequence of finite hulls $\{\mathcal{M}_i\colon i \in \omega\}$ can be constructed for a certain finite set of formulas which is yet to be chosen but which will not depend on the particular choice of $\{\mathcal{M}_i\colon i \in \omega\}$ which we have made. Then as the proof proceeds, whenever we encounter a true sentence which begins with a bounded existential quantifier $(\exists x)$ and involving parameters in $\mathcal{M}_i$, we can change the bounded existential quantifier to $(\exists x \in \mathcal{M}_{i+1})$ and still have a true sentence. That is, if we have a sentence of the form $(\exists x)\ \phi(x, v_0, v_1, \ldots, v_n)$ where x and the v_i's are free variables and there is some x and $m_0, \ldots, m_n \in \mathcal{M}_i$ such that $\phi(x, m_0, \ldots, m_n)$, then there is also some $x \in \mathcal{M}_{i+1}$ such that $\phi(x, m_0, \ldots, m_n)$. At the end of the proof, we simply look at the finitely many formulas to which we applied this process and pretend that we chose them at the outset before $\{\mathcal{M}_i\colon i \in \omega\}$ was constructed. As with elementary submodels, any application of finite hulls can be removed by simply replacing it with the particular instance of the combinatorial construction given above. This is exactly why any application of finite hulls to topology is formally unnecessary.

Choosing a large natural number n (or even $n = 6$), one gains strength by using the sequence $\{\mathcal{M}_{ni}\colon i \in \omega\}$. A more general approach would be to work in a non-standard model and use a nonstandard natural number n.

Many arguments in topology are essentially finite hull arguments. For example, the proof by Eric van Douwen of the existence of a remote point and all its derivatives such as Dow's work on endowments are basically finite hull arguments.

Theorem 2. *Let $\mathcal{I} \subset \mathcal{P}(\kappa)$. The following are equivalent:*

(1) *There is a topological space X with a discrete family of points κ such that $\{A \subset \kappa\colon A \text{ is separated}\} = \mathcal{I}$.*

(2) *There is a T_1 space X with a discrete family of points κ such that $\{A \subset \kappa\colon A \text{ is separated}\} = \mathcal{I}$.*

(3) *$\mathcal{I}$ is diagonally-closed and contains all singletons.*

Proof. (1) $\Rightarrow$ (3) Suppose $A \subset \kappa$, $\{E_i\colon i \in \omega\}$ and $\{B_i\colon i \in \omega\}$ are as in Definition 1. We shall define a neighborhood assignment $\{N_\alpha\colon \alpha \in A\}$ so that $\alpha \neq \alpha' \Rightarrow N_\alpha \cap N_{\alpha'} = \emptyset$. Fix $\alpha \in A$. Find $i_\alpha \in \omega$ such that $\alpha \in B_{i_\alpha}$. Choose N_α so that whenever $C \in E_{i_\alpha}$ and $\alpha \in C$, then $\{N_\alpha\colon \alpha \in C\}$ witnesses that C is separated.

This neighborhood assignment shows that A is separated (and thus that $A \in \mathcal{I}$ as required. To see this, suppose $\beta, \gamma \in A$. Suppose $\beta \in B_i$, $\gamma \in B_k$ and $i \leqslant k$. Find $C \in E_i$ so that $\{\beta, \gamma\} \subset C$. Now both N_β and N_γ are disjoint.

(3) $\Rightarrow$ (2) Suppose $\mathcal{I}$ is diagonally-closed and contains all singletons. Let

$$\mathrm{ISO} = \big\{\{(\alpha, V), (\beta, W)\} \in [\kappa \times [\mathcal{I}]^{<\omega}]^2\colon\ (\forall P \in V \cap W)\ \{\alpha, \beta\} \not\subset P\big\}.$$

Topologize $X = \kappa \cup \mathrm{ISO}$ by letting ISO consist of isolated points and letting a neighborhood $\alpha(U)$ of $\alpha \in \kappa$ in parameter $U \in [\mathcal{I}]^{<\omega}$ be

$$\{\alpha\} \cup \big\{\{(\alpha, V), (\beta, W)\}\colon\ V \supset U\big\}.$$

Since $U \subset U'$ implies $\alpha(U) \supset \alpha(U')$ and U can be an arbitrary finite subset of $\mathcal{I}$ and since the intersection of $\alpha(U)$ and $\alpha'(U')$ consists of isolated points when $\alpha \neq \alpha'$, we know that this defines a topological space. To show that X is T_1, it suffices to note that $\bigcap\{\alpha(U)\colon\ U \in [\mathcal{I}]^{<\omega}\} = \{\alpha\}$.

The family of open sets $\{\alpha(\{A\})\colon\ \alpha \in A\}$ separates A when $A \in \mathcal{I}$. Again, the hard part of the proof is to show that, whenever A can be separated, we must have $A \in \mathcal{I}$. So suppose $A \subset \kappa$ and that $\{U_\alpha\colon\ \alpha \in A\}$ have been defined so that $\{\alpha(U_\alpha)\colon\ \alpha \in A\}$ is a disjoint family of open sets. We shall show that $A \in \mathcal{I}$.

Let $\Omega(\alpha) = \{P \in U_\alpha\colon\ \alpha \in P\}$. If there are $\alpha, \beta \in A$ so that $\Omega(\alpha) \cap \Omega(\beta) = \emptyset$, then $\{(\alpha, U_\alpha), (\beta, U_\beta)\} \in \alpha(U_\alpha) \cap \beta(U_\beta)$ which is impossible.

So we know that $\Omega : A \to [\mathcal{I}]^{<\omega}$ is a function such that $(\forall \alpha \in A)(\forall P \in \Omega(\alpha))\ \alpha \in P$ and $(\forall \alpha, \beta \in A)\ \Omega(\alpha) \cap \Omega(\beta) \neq \emptyset$. Note that we may assume $|A| > 1$ since $\mathcal{I}$ already contains all the singletons. Note also that each $\Omega(\alpha)$ is nonempty.

Let $\{\mathcal{M}_n\colon\ n \in \omega\}$ be an increasing sequence of finite hulls where $\Omega, A \in \mathcal{M}_0$. Let

$$B_i = \Big\{\gamma \in A - \bigcup\{B_j\colon\ j < i\}\colon\ (\forall \beta \in A)\ \Omega(\beta) \cap \Omega(\gamma) \cap \mathcal{M}_i \neq \emptyset\Big\}.$$

We claim that $\bigcup\{B_i\colon\ i \in \omega\} = A$. Suppose otherwise that $\beta \in A$ but $(\forall i \in \omega)\beta \notin B_i$. Now fix $i \in \omega$. Since $\beta \notin B_i$, there is $\alpha \in A$ such that $\Omega(\alpha) \cap \Omega(\beta) \cap \mathcal{M}_i = \emptyset$. Now let $N = \mathcal{M}_i \cap \Omega(\alpha)$. So we have established

$$N \cap \Omega(\beta) = \emptyset \tag{1}$$

Now $N \in \mathcal{M}_{i+2}$, by Proposition 3, and $(\exists \alpha \in A)\ \mathcal{M}_i \cap \Omega(\alpha) = N$. Thus, by reflection in Proposition 1

$$(\exists \alpha_0 \in A \cap \mathcal{M}_{i+3})\ \mathcal{M}_i \cap \Omega(\alpha_0) = N \tag{2}$$

Choose $P \in \Omega(\alpha_0) \cap \Omega(\beta)$. Now $P \notin N$ by Eq. (1) and $P \notin \mathcal{M}_i$ by Eq. (2). But $\alpha_0 \in \mathcal{M}_{i+3}$ and so, by Proposition 2, $\Omega(\alpha_0) \in \mathcal{M}_{i+4}$.[2] Since $\Omega(\alpha_0)$ is finite, $\Omega(\alpha_0) \subset \mathcal{M}_{i+4}$, by finitary transitivity in proposition 1, and so $P \in \mathcal{M}_{i+4}$. So $\Omega(\beta) \cap (\mathcal{M}_{i+4} - \mathcal{M}_i) \neq \emptyset$. Thus $\Omega(\beta)$ is infinite since i is arbitrary. This is a contradiction which proves the claim that $\bigcup\{B_i\colon\ i \in \omega\} = A$.

[2] Since $\Omega(\alpha_0)$ is uniquely defined from Ω and α_0.

The definition of diagonally closed can be applied for this A, these B_i and $E_i = \mathcal{M}_i \cap \mathcal{I}$. To see this, take $i \in \omega$, $\beta \in B_i$ and $\gamma \in A$ and choose $C \in \Omega(\gamma) \cap \Omega(\beta) \cap \mathcal{M}_i$ by the definition of B_i. Since $\gamma \in C$, $\beta \in C$, we have $\{\beta, \gamma\} \subset C$ as required.

We have shown that $A \in \mathcal{I}$. □

The property of being diagonally-closed may seem to be excessively complicated and the reader may speculate that a simpler property might suffice. But we must keep in mind two kinds of operations (which do not seem to admit simplifications) under which the family of separated sets must be closed.

The first of these is quite simple. Whenever A is partitioned into three sets A_0, A_1, A_2 so that $A_0 \cup A_1$, $A_0 \cup A_2$ and $A_1 \cup A_2$ are separated, then A is separated. This is the special case of diagonally-closed when $B_0 = A$ and $E_0 = \{A_0 \cup A_1, A_0 \cup A_2, A_1 \cup A_2\}$. The second of these is more complex. Suppose $A = \{f \in 3^\omega\colon |f^{-1}(0)| \neq \emptyset\}$. Suppose that, for each $n \in \omega$ and $i \in \{1, 2\}$, $\{f\colon f(n) \in \{0, i\}\}$ is separated. A standard induction shows that A is also separated.

This is just the special case of diagonally closed in which $B_n = \{f\colon \min f^{-1}(0) = n\}$ and $E_n = \{\{f\colon f(m) \in \{0, i\}\}\colon i \in \{1, 2\},\ m \leqslant n\}$.

When the reader notes the kinds of combinations of these two operations which lead to separated sets, the definition of diagonally-closed emerges as quite natural and it begins to seem likely that no simpler property than diagonally-closed can be found.

Theorem 3. *Let $\mathcal{I} \subset \mathcal{P}(\kappa)$. There is a Hausdorff space X with a discrete family of points κ such that $\{A \subset \kappa\colon A \text{ is separated}\} = \mathcal{I}$ if and only if $\mathcal{I}$ is diagonally-closed and contains all sets of size two.*

Proof. If $\mathcal{I} \supset [\kappa]^2$, then the space constructed in Theorem 2 is a Hausdorff space. □

Since any diagonally-closed family containing all sets of size two also contains all finite sets, we can replace "sets of size two" in the characterization in Theorem 3 by "finite sets".

Corollary 2 (Hajnal and Juhasz [2]). *There is a Hausdorff space with an infinite closed discrete set in which no infinite subset can be separated.*

Proof. $[\kappa]^{<\omega}$ is diagonally-closed. □

Of course, Theorems 1, 2, 3 and 4 hold when κ is finite. But in all the theorems except Theorem 2, the family $\mathcal{I}$ must be all of $\mathcal{P}(\kappa)$. In Theorem 2, the condition that $\mathcal{I}$ is diagonally-closed reduces to the simpler property that there is a finite subset M of $\mathcal{I}$ such that $(\forall \beta, \gamma \in A)(\exists C \in M)\ \{\beta, \gamma\} \subset C$.

Definition 2. If $\mathcal{I} \subset \mathcal{P}(\kappa)$, then we say $\mathcal{I}$ is *weakly-countably-closed* if, whenever $A \in \mathcal{I}$, $B \in [\kappa]^\omega$ and $\{A_i\colon i \in \omega\}$ is a partition of A such that, for each $i \in \omega$, $A_i \cup B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

If $\mathcal{I} \subset \mathcal{P}(\kappa)$, then we say $\mathcal{I}$ is *closed under adding a finite set* if, whenever $A \in \mathcal{I}$, $B \in [\kappa]^{<\omega}$, then $A \cup B \in \mathcal{I}$.

Proposition 4. *If $\mathcal{I}$ is diagonally closed and closed under adding a finite set, then $\mathcal{I}$ is weakly-countably-closed.*

Proof. Let $B = \{\beta_i\colon\ i \in \omega\}$. The partition is $\{A_i \cup \{\beta_i\}\colon\ i \in \omega\}$. The increasing finite subsets of $\mathcal{I}$ are $E_n = \{B \cup A_i\colon\ i \leqslant n\} \cup \{A \cup \{\beta_j\colon\ j \leqslant i\}\colon\ i \leqslant n\}$. □

Theorem 4. *Let $\mathcal{I} \subset \mathcal{P}(\kappa)$. The following are equivalent:*

(1) *There is a regular space X with a discrete family of points κ such that $\{A \subset \kappa\colon\ A$ is separated$\} = \mathcal{I}$.*

(2) *There is a completely regular space X with a discrete family of points κ such that $\{A \subset \kappa\colon\ A$ is separated$\} = \mathcal{I}$.*

(3) *$\mathcal{I}$ is diagonally-closed, closed under adding a finite set, and contains all countable sets.*

Proof. (1) $\Rightarrow$ (3) This is immediate since, in a regular space, countable sets are separated and adding a finite set to a separated set produces a separated set.

(3) $\Rightarrow$ (2) Let

$$\begin{aligned}\mathrm{ISO} = \big\{\{(\alpha, G, V), (\beta, H, W)\} \in \big[\kappa \times [\kappa]^{<\omega} \times [\mathcal{I}]^{<\omega}\big]^2\colon \\ \alpha \notin H,\ \beta \notin G,\ (\forall P \in V \cap W)\ \{\alpha, \beta\} \not\subset P\big\}.\end{aligned}$$

Topologize $X = \kappa \cup \mathrm{ISO}$ by letting ISO consist of isolated points and letting a neighborhood $\alpha(F, U)$ of $\alpha \in \kappa$ in parameters $F \in [\kappa]^{<\omega}$ and $U \in [\mathcal{I}]^{<\omega}$ be

$$\{\alpha\} \cup \big\{\{(\alpha, G, V), (\beta, H, W)\}\colon\ G \supset F,\ V \supset U\big\}.$$

We have defined a T_1 topological space for the same reasons as in Theorems 1 and 2. To show that X is completely regular, we show that X is zero-dimensional. We do this by noting that, if $\gamma \in \kappa$ and $\gamma \neq \alpha$, then $\gamma(\{\alpha\}, \emptyset) \cap \alpha(F, U) = \emptyset$.

The family of open sets $\{\alpha(\emptyset, \{A\})\colon\ \alpha \in A\}$ separates A when $A \in \mathcal{I}$. Once again, the hard part of the proof is to show that, if A can be separated, then $A \in \mathcal{I}$. So suppose $A \subset \kappa$ and that $\Omega\colon A \to [\mathcal{I}]^{<\omega}$ and $\Theta\colon A \to [\kappa]^{<\omega}$ are functions so that $\{\alpha(\Theta(\alpha), \Omega(\alpha))\colon\ \alpha \in A\}$ is a disjoint family of open sets. We shall show that $A \in \mathcal{I}$.

We can assume $(\forall \alpha \in A)(\forall P \in \Omega(\alpha))\ \alpha \in P$ and $(\forall \alpha \in A)\Theta(\alpha) \subset A$ since any elements of $\Omega(\alpha)$ which did not contain α and any elements of $\Theta(\alpha)$ which are not in A would have no effect on whether any two of the $\alpha(\Theta(\alpha), \Omega(\alpha))$'s intersect or not. If $\alpha(\Theta(\alpha), \Omega(\alpha))$ and $\alpha'(\Theta(\alpha'), \Omega(\alpha'))$ are disjoint, then by inspecting the definition of these open sets, we can calculate that $(\forall \alpha, \beta \in A)(\Omega(\alpha) \cap \Omega(\beta) \neq \emptyset$ or $\alpha \in \Theta(\beta)$ or $\beta \in \Theta(\alpha))$.

Define $\{\mathcal{M}_n\colon\ n \in \omega\}$ with $A, \Omega, \Theta \in \mathcal{M}_0$. Let $\mathcal{M}_\omega = \bigcup\{\mathcal{M}_n\colon\ n \in \omega\}$. Let $B_i = \{\gamma \in A - \bigcup\{B_j\colon\ j < i\}\colon\ (\forall \beta \in A - \mathcal{M}_\omega)\Omega(\beta) \cap \Omega(\gamma) \cap \mathcal{M}_i \neq \emptyset\}$.

We claim that $\bigcup\{B_i\colon\ i \in \omega\} \supset A - \mathcal{M}_\omega$

Suppose otherwise that $\beta \in A - \mathcal{M}_\omega$ but, for all $i \in \omega$, $\beta \notin B_i$. Fix $i \in \omega$. If $\beta \notin B_i$ then there is $\alpha^* \in A - \mathcal{M}_\omega$ such that $\Omega(\alpha^*) \cap \Omega(\beta) \cap \mathcal{M}_i = \emptyset$. Let $N = \mathcal{M}_i \cap \Omega(\alpha^*)$ so

$$N \cap \Omega(\beta) = \emptyset \tag{3}$$

Now $N \in \mathcal{M}_{i+2}$ by Proposition 3. Choose $\alpha_0 \in A \cap \mathcal{M}_\omega$, if possible, so that

$$\mathcal{M}_i \cap \Omega(\alpha_0) = N \tag{4}$$

and $\alpha_0 \notin \Theta(\beta)$.

If we did not choose α_0, then $R = \{\alpha \in A \cap \mathcal{M}_\omega\colon\ \mathcal{M}_i \cap \Omega(\alpha) = N\}$ is finite. So there is $n > i$ such that $R \subset \mathcal{M}_n$. Thus $R \in \mathcal{M}_{n+2}$. by Proposition 3. We claim

$$R = \{\alpha \in A\colon\ \mathcal{M}_i \cap \Omega(\alpha) = N\}.$$

If not, then there is $\alpha \in A - R$ such that $\mathcal{M}_i \cap \Omega(\alpha) = N$. Thus there is, by reflection, $\alpha' \in \mathcal{M}_{n+3} \subset \mathcal{M}_\omega$ such that $\mathcal{M}_i \cap \Omega(\alpha') = N$ and $\alpha' \in A - R$ which contradicts the definition of R. Now $\alpha^* \in R \subset \mathcal{M}_{n+2} \subset \mathcal{M}_\omega$ (by finite transitivity) which contradicts the choice of α^*.

So α_0 is indeed chosen. Now since $\alpha_0 \in \mathcal{M}_\omega$, there is n so that $\alpha_0 \in \mathcal{M}_n$ and so $\Theta(\alpha_0) \in \mathcal{M}_{n+1}$ and $\Omega(\alpha_0) \in \mathcal{M}_{n+1}$ by Proposition 2.[3] Thus, by finitary transitivity, $\Theta(\alpha_0) \subset \mathcal{M}_{n+1} \subset \mathcal{M}_\omega$ and also $\Omega(\alpha_0) \subset \mathcal{M}_{n+1} \subset \mathcal{M}_\omega$.

Now $\Omega(\alpha_0) \cap \Omega(\beta) \neq \emptyset$ or $\alpha_0 \in \Theta(\beta)$ or $\beta \in \Theta(\alpha_0)$. If $\beta \in \Theta(\alpha_0)$, then $\beta \in \Theta(\alpha_0) \subset \mathcal{M}_\omega$ which is impossible. If $\alpha_0 \in \Theta(\beta)$, then the choice of α_0 is incorrect.

We can therefore deduce that $\Omega(\alpha_0) \cap \Omega(\beta) \neq \emptyset$. Choose $P \in \Omega(\alpha_0) \cap \Omega(\beta)$. Now $P \notin N$ by Eq. (3) and $P \notin \mathcal{M}_i$ by Eq. (4). Thus $P \in \Omega(\alpha_0) \subset \mathcal{M}_\omega$ and so $\Omega(\beta) \cap (\mathcal{M}_\omega - \mathcal{M}_i) \neq \emptyset$. Since i was arbitrary, $\Omega(\beta)$ was infinite which is impossible.

We have proved the claim that $\bigcup\{B_i\colon\ i \in \omega\} \supset A - \mathcal{M}_\omega$.

Now the definition of diagonally-closed applies where E_i takes on the value $\mathcal{M}_i \cap \mathcal{I}$, A takes on the value $A - \mathcal{M}_\omega$ and B_i takes on the value $B_i - \mathcal{M}_\omega$. To see this, suppose $i \in \omega$, $\beta \in B_i - \mathcal{M}_\omega$ and $\gamma \in A - \mathcal{M}_\omega$. By the definition of B_i, $\Omega(\beta) \cap \Omega(\gamma) \cap \mathcal{M}_i \neq \emptyset$ which implies that there is $C \in \mathcal{M}_i \cap \mathcal{I}$ such that $C \in \Omega(\beta) \cap \Omega(\gamma)$ and so $\{\beta, \gamma\} \subset C$. Thus we deduce that $A - \mathcal{M}_\omega \in \mathcal{I}$.

Now let $B = A \cap \mathcal{M}_\omega = \{\alpha_n\colon\ n \in \omega\}$. We also know that $A \cap \mathcal{M}_\omega \in \mathcal{I}$ since $\mathcal{I}$ contains all countable sets. We know that $(\forall \alpha \in A - \mathcal{M}_\omega)(\exists n_\alpha \in \omega)\ \Theta(\alpha) \cap A \cap \mathcal{M}_\omega \subset \{\alpha_n\colon\ n \leqslant n_\alpha\}$. Let $A_n = \{\alpha \in A - \mathcal{M}_\omega\colon\ n_\alpha = n\}$. Now $\alpha_n \in \mathcal{M}_\omega$ implies that there is k such that $\alpha_n \in \mathcal{M}_k$. But now $\Theta(\alpha_n) \in \mathcal{M}_{k+1}$, by Proposition 2, and $\Theta(\alpha_n) \subset \mathcal{M}_{k+1}$ by finitary transitivity. Thus $(\forall n \in \omega)\Theta(\alpha_n) \subset A \cap \mathcal{M}_\omega$.

Now if $\alpha \in A_n$ and $m > n$, then $\alpha_m \notin \Theta(\alpha)$ (since $n < m$) and $\alpha \notin \Theta(\alpha_m)$ (since $\alpha \notin \mathcal{M}_\omega$). So we have $\Omega(\alpha) \cap \Omega(\alpha_m) \neq \emptyset$.

For all $\alpha \in A_n$, find m_α so that

$$\Omega(\alpha) \cap \bigcup\{\Omega(\alpha_i)\colon\ i \in \omega\} = \Omega(\alpha) \cap \bigcup\{\Omega(\alpha_i)\colon\ i \leqslant m_\alpha\}.$$

[3] Since each of these can be uniquely defined from Θ, Ω and α_0.

Let $A_n^m = \{\alpha \in A_n\colon\ m = m_\alpha\}$. Apply diagonally closed for $B_m = A_n^m \cup \{\alpha_{m+n+1}\}$ and

$$E_m = \bigcup \{\Omega(\alpha_i)\colon\ i \leqslant m\} \cup \{\{\alpha_m\colon\ m \in \omega\}\} \cup \{(A_n \cup \{\alpha_{m+n+1}\})\}$$

and $A = A_n \cup \{\alpha_m\colon\ m > n\}$. To see that we can do this, fix $m \in \omega$. Choose $\beta \in A_n^m \cup \{\alpha_{m+n+1}\}$ and $\gamma \in A_n \cup \{\alpha_m\colon\ m > n\}$. If $\beta \in A_n^m$ and $\gamma = \alpha_{m'}$ for some $m' > n$, then since $\Omega(\beta) \cap \Omega(\alpha_{m'}) \neq \emptyset$, there is $C \in \Omega(\beta) \cap \Omega(\alpha_{m'})$ and now $\{\beta, \alpha_{m'}\} \subset C$. But $C \in \Omega(\alpha_{m''})$ for some $m'' \leqslant m$ since $\beta \in A_n^m$.

If $\beta = \alpha_{m+n+1}$ and $\gamma = \alpha_m$ for some $m > n$, then $C = \{\alpha_m\colon\ m \in \omega\}$ works. If $\beta \in A_n^m \cup \{\alpha_{m+n+1}\}$ and $\gamma \in A_n$, then $C = A_n \cup \{\alpha_{m+n+1}\}$ works.

Thus $A_n \cup \{\alpha_m\colon\ m > n\} \in \mathcal{I}$.

Since $\mathcal{I}$ is invariant under adding a finite set, this implies that $A_n \cup B \in \mathcal{I}$. Now we can apply Proposition 4 to get $A \in \mathcal{I}$ as required. □

Corollary 3. *There is a regular space with an uncountable closed discrete set in which any two disjoint infinite closed discrete sets can be separated if and only if both are countable.*

Proof. Let $\mathcal{I}$ be the family of all countable sets. Apply the construction of Theorem 4. □

We have given satisfactory internal characterizations of the family of separated subsets of a topological, Hausdorff and regular space. That is, we have removed the leading existential quantifier in the definition of a family of separated subsets in some topological, Hausdorff or regular space.

A corollary to the proofs of these results is the existence of natural test spaces for any family.

Proposition 5. *There is, for each family $\mathcal{I} \subset \mathcal{P}(\kappa)$, a topological (Hausdorff, regular, normal) space $X(\mathcal{I})$ with a closed discrete subset κ so that, if $\mathcal{J}$ is the family of separated subsets of $X(\mathcal{I})$, then $\mathcal{J}$ is the smallest family of subsets of κ containing $\mathcal{I}$ which is the family of separated subsets of κ in some topological (Hausdorff, regular, normal) space in which κ is a closed discrete subset.*

Can we identify families which can be the family of separated sets in a Hausdorff space but not in a regular space? Of course, any family which can be the family of separated subsets in a Hausdorff space but which does not contain all countable sets or which is not closed under the addition of finite sets cannot be the family of separated subsets in a regular space, but can there be others?

Proposition 6. *There is a family of subsets of a set of size $\aleph_1$ which can be the family of separated subsets of a Hausdorff space, contains all countable sets and is closed under adding finite sets but which cannot be the family of separated subsets of a regular space.*

Proof. Let $X = (\omega_1 \times \omega) \cup \omega$. Let $\mathcal{J}$ be the set consisting of all countable subsets of X, each $(\omega_1 \times \{n\}) \cup \omega$ and $\omega_1 \times \omega$. Let $\mathcal{I}$ be the result of closing $\mathcal{J}$ under the taking of subsets, the adding of finite sets and the taking of the diagonal-closure. Let $\mathcal{K}$ be the closure of $\mathcal{I}$ under weak-countable-closure. Note that $X \in \mathcal{I}$ and yet $X \notin \mathcal{K}$. Thus $\mathcal{I}$ is the family of separated subsets of some Hausdorff space, $\mathcal{I}$ is closed under the adding of finite sets and contains all countable sets and yet $\mathcal{I}$ is not the family of separated subsets of a regular space. $\square$

Problem 2. Characterize those $\mathcal{I} \subset \mathcal{P}(\kappa)$ which can be the family of separated sets in a δ-normal space (a T_1 space in which countable sets can be separated from closed sets) in which κ is a closed discrete subset.

Problem 3. Characterize those $\mathcal{I} \subset \mathcal{P}(\kappa)$ for which there is a topological (Hausdorff) (regular) space, in which κ is a closed discrete subset, such that $A \in \mathcal{I}$ iff A and $\kappa - A$ can be separated by disjoint open sets

Of course, the spaces of Theorems 1 and 4 are resolutions of the space of Theorem 3 (see [5]). In each case, we have "coded in" an additional property (normality and regularity, respectively). This method of coding was developed in [4].

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